NASA TECHNICAL TRANSLATION NASA TT F-15,810

OPTIMAL TRANSFER BETWEEN COPLANAR ELLIPTICAL ORBITS USING TANGENTIAL PULSES APPLIED AT APSIDAL POINTS

A. S. Shmyrov

(NASA-TT-F-15810) OPTIMAL TRANSFER BETWEEN COPLANAR ELLIPTICAL ORBITS USING TANGENTIAL PULSES APPLIED AT APSIDAL POINTS (Kanner (Leo) Associates) 13 p N74-33289
CSCL 03C G3/30 Unclas 48743

Translation of "Optimal'nyy perekhod mezhdu komplanarnymi ellipticheskimi orbitami s pomoshch'yu kasatel'nykh impul'sov prilozhennykh v apsidal'nykh tochkakh," in: Mekhanika upravlyayemogo dvizheniya i problemy kosmicheskoy dinamiki, Ed. by V. S. Novoselov, Leningrad, University Press, Leningrad, 1972, pp. 63-69.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D. C. 20546 AUGUST 1974

13

STANDARD TITLE PAGE

1. Report No. NASA TT F-15810	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle OPTIMAL TRANSFER BETWEEN COPLANAR ELLIPTICAL ORBITS USING TANGENTIAL PULSES APPLIED AT APSIDAL POINTS		5. Report Date August 1974
		6. Performing Organization Code
7. Author(s) A. S. Shmyrov		8. Performing Organization Report No.
		10. Work Unit No.
9. Performing Organization Name and Address Leo Kanner Associates Redwood City, California 94063		11. Contract or Grant No. NASw-2481-407
		13. Type of Report and Period Covered Translation
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration, Washington, D. C. 20546		14. Sponsoring Agency Code

15. Supplementary Notes

Translation of "Optimal'nyy perekhod mezhdu komplanarnymi ellipticheskimi orbitami s pomoshch'yu kasatel'nykh impul'sov prilozhennykh v apsidal'nykh tochkakh," in: Mekhanika upravlyayemogo dvizheniya i problemy kosmicheskoy dinamiki, Leningrad University Press, Leningrad, 1972, pp. 63-69.

16. Abstract

Numerical methods and numerous analytic studies have been made to solve the problem of optimal pulse transfer between orbits. This study provides a solution of the particular case of the general problem using sufficient conditions of optimacy. The particular case constitutes coplanar elliptical orbits whose apsidal lines coincide.

17. Key Words (Selected by Author(s))	18. Distribution Statement Unclassified-Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified	11	

OPTIMAL TRANSFER BETWEEN COPLANAR ELLIPTICAL ORBITS USING TANGENTIAL PULSES APPLIED AT APSIDAL POINTS

A. S. Shmyrov

The problem of optimal pulse transfer between orbits is a /63 vital problem of celestial mechanics. Numerical methods and numerous analytic studies have been made toward its solution [1]. There are, moreover, publications on numerical analysis of optimal trajectories of acquisition of actual celestial bodies. This study provides a solution of the particular case of the general problem using sufficient conditions of optimacy.

1. Statement of the Problem

Let us consider the set of coplanar elliptical orbits, whose apsidal lines coincide. Let us select a direction on the apsidal line and call the apsidal point of the orbit positive, if its directional radius-vector is positive; the other apsidal point will, accordingly, be negative. The distances from the center of gravity to the apsidal points will be denoted by r_+ and r_-. The elements /64 of the orbit become fully defined with the assignment of two positive numbers r_+ and r_-. In particular, by these values we can calculate the value of velocities v_+ and v_- of the apsidal points:

$$\begin{aligned} v_+ &= (2\mu)^{1/2} r_-^{1/2} r_+^{-1/2} (r_+ + r_-)^{-1/2}, \\ v_- &= (2\mu)^{1/2} r_-^{-1/2} r_+^{1/2} (r_+ + r_-)^{-1/2}, \end{aligned} \quad (1.1)$$

where μ is the constant of gravity.

Now let it be required to move between two orbits from the set in question using tangenital pulses applied at apsidal points so that the characteristic velocity of transfer is minimal. Formula

(1.1) permits us to establish how the values r_+ and r_- vary as a function of increases in velocity. Given that at a positive apsidal point velocity has received an infinitely small increase $dv > 0$. The quantity r_+ herein will not change, but the increase dr_- of quantity r_- can be calculated from the first formula of (1.1):

$$dr_- = \pm(\mu/2)^{-1/2} r_+^{-1/2} r_-^{1/2} (r_+ + r_-)^{3/2} dv, \tag{1.2}$$

where the sign 'plus' is selected if the increase is given in the direction of velocity, and the sign 'minus' if otherwise.

If, however, the increase dv was received by the quantity v_-, the value r_- will not change, but the increase dr_+ is defined from the second formula of (1.1):

$$dr_+ = \pm(\mu/2)^{-1/2} r_+^{1/2} r_-^{-1/2} (r_+ + r_-)^{3/2} dv. \tag{1.3}$$

The sign on the right side of (1.3) is selected just as in (1.2).

Therefore, the derivatives of r_+', r_-' of the quantities r_+ and r_- in terms of characteristic velocity v depend on the point of application of the pulse and its direction with respect to velocity. They generally appear as

$$\begin{aligned} r_+' &= (\mu/2)^{-1/2} r_+^{1/2} r_-^{-1/2} (r_+ + r_-)^{3/2} u_1 u_2, \\ r_-' &= (\mu/2)^{-1/2} r_+^{-1/2} r_-^{1/2} (r_+ + r_-)^{3/2} u_1 (1 - u_2), \end{aligned} \tag{1.4}$$

where $u_1 = \pm 1$, $u_2 = 0;\ 1$.

The quantities u_1, u_2 are controls and define the applied pulse as follows: if $u_1 = 1$, the pulse is applied in the direction of velocity; if $u_2 = 0$, the pulse is applied at the positive apsidal point; if $u_2 = 1$, the pulse is applied at the negative apsidal point; if $u_1 = -1$, the pulse is aimed opposite velocity.

Now this problem can be formulated as a problem on the optimal speed of response: to select such controls u_1 and u_2 so the initial point (r_+^i, r_-^i) transfer to a final point (r_+^f, r_-^f) with the least increase in the argument.

2. Description of Control Synthesis

/65

Let us introduce instead of the variables r_+, r_- new variables x_1, x_2, t according to the formulas

$$x_1 = r_+^{-1}, \quad x_2 = r_-^{-1}, \quad t = (\mu/2)^{-1/2} v.$$

Equations (1.4) in the new variables will acquire the appearance

$$\begin{aligned} \dot{x}_1 &= x_2^{-1}(x_1 + x_2)^{3/2} u_1 u_2, \\ \dot{x}_2 &= x_1^{-1}(x_1 + x_2)^{3/2} u_1 (1 - u_2), \end{aligned} \tag{2.1}$$

where $(\cdot)$ denotes the differentiation with respect to t.

Let us limit the set of possible values of x_1, x_2 by the square of A: $a_1 \leqq x_1 \leqq a_2$, $a_1 \leqq x_2 \leqq a_2$, where $a_1 << 1$, $a_2 >> 1$.

The meaning of this limitation is that the initial, final and transitional orbits lie within a circle formed by neighborhoods with radii a_1^{-1}, a_2^{-1}.

Let us now examine the problem of optimal incidence from any point of the square of A into point (1, k) for determinacy positing that $k > 1$. This problem is considered solved if controls u_1, u_2 are found as functions of the variables x_1 and x_2, i.e., if synthesis of the optimal control is made.

Let us first give a description of the synthesis, and the proof of its optimacy will be made later.

In a figure is depicted the subdivision of square A into areas A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9 by curves MM_1, M_1M_2, MM_3, M_2M_3, M_3M_4, M_4M_5, M_5M_8, M_4M_6, M_6M_7, M_7M_8, M_7M_9, MM_8, M_8M_9, M_9M_{11}, $M_{10}M_{11}$, $M_{10}M_{12}$, M_1M_{12}, MM_{10}, $M_{12}M_{13}$, $M_{13}M_{14}$. Each of these curves is defined by an equation and inequality which have a form indicated below.

For curve M_1M_2

$$(x_2-x_1)(x_1+x_2)^{-1/2}-(x_2-a_1)(x_2+a_1)^{-1/2}+(x_1+k)^{1/2}=(k+a_1)^{1/2}, \quad a_1\leqslant x_1\leqslant 1. \tag{2.2}$$

For curve MM_1

$$x_1=1, \quad k\leqslant x_2\leqslant x_{21}, \tag{2.3}$$

where x_{21} is the solution of equation (2.2) with respect to x_2 where $x_1 = 1$.

For curve M_2M_3

$$x_1=a_1, \quad k\leqslant x_2\leqslant x_{22}, \tag{2.4}$$

where x_{22} is the solution with respect to x_2 of equation (2.2) where $x_1 = a_1$.

For curve MM_3

$$x_2=k, \quad a_1\leqslant x_1\leqslant 1. \tag{2.5}$$

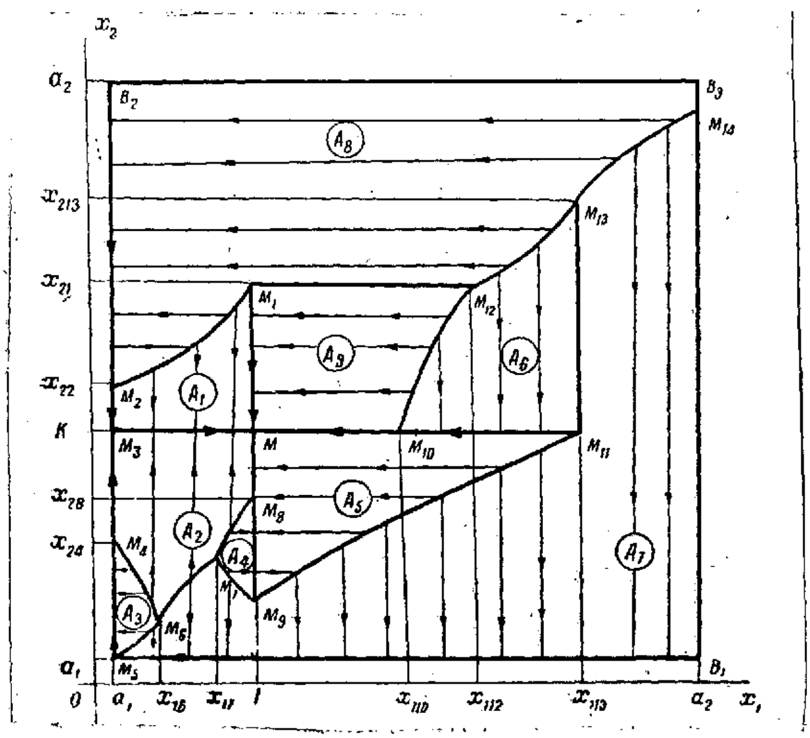
For curve M_4M_6

$$(x_1+x_2)^{1/2}+(k-x_1)(k+x_1)^{-1/2}-x_2(x_1+a_1)^{-1/2}=k(k+a_1)^{-1/2}, \quad a_1\leqslant x_1\leqslant x_{16}. \tag{2.6}$$

For curve M_5M_6 /66

$$(x_1-x_2)(x_1+x_2)^{-1/2}+(x_2+a_1)^{1/2}-(x_1-a_1)(x_1+a_1)^{-1/2}=(2a_1)^{1/2}, \quad a_1\leqslant x_1\leqslant x_{16}, \tag{2.7}$$

where x_{16} is the solution with respect to x_1 of the system of equations (2.6) and (2.7) in square A.



For curve M_6M_7

$$2x_1(x_1+x_2)^{-1/2}+(k-x_1)(k+x_1)^{-1/2}-(x_1-a_1)(x_1+a_1)^{-1/2}=(2a_1)^{1/2}+(k-a_1)(k+a_1)^{-1/2}+k(k+1)^{-1/2}, \quad x_{16}\leqslant x_1\leqslant x_{17}. \tag{2.8}$$

For curve M_7M_8

$$(x_1-x_2)(x_1+x_2)^{-1/2}+(x_2-1)(x_2+1)^{-1/2}+(k-x_1)(k+x_1)^{-1/2}=$$
$$=(k-1)(k+1)^{-1/2}, \quad x_{17}\leqslant x_1\leqslant 1, \tag{2.9}$$

where x_{17} is the solution with respect to x_1 of system of equations (2.8) and (2.9).

For curve M_7M_9 /67

$$(x_1+x_2)^{1/2}+(1-x_2)(1+x_2)^{-1/2}-(x_1-a_1)(x_1+a_1)^{-1/2}=$$
$$=(2a_1)^{1/2}+(k-a_1)(k+a_1)^{-1/2}+(1-k)(1+k)^{-1/2}, \quad x_{17}\leqslant x_1\leqslant 1. \tag{2.10}$$

For curve M_9M_{11}

$$(x_2-x_1)(x_2+x_1)^{-1/2}+(x_2+1)^{1/2}-(x_1-a_1)(x_1+a_1)^{-1/2}=$$
$$=(2a_1)^{1/2}+(k-a_1)(k+a_1)^{-1/2}-(k-1)(k+1)^{-1/2}, \quad x_{29}\leqslant x_2\leqslant k, \tag{2.11}$$

where x_{29} is the solution with respect to x_2 of equation (2.11) where $x_1 = 1$.

For curve M_8M_9

$$x_1=1, \quad x_{29}\leqslant x_2\leqslant x_{28}. \tag{2:12}$$

For curve MM_8

$$x_1=1, \quad x_{28}\leqslant x\leqslant k, \tag{2.13}$$

where x_{28} is the solution of equation (2.9) where $x_1 = 1$ with respect to x_2.

For curve $M_{10}M_{12}$

$$(x_1-x_2)(x_1+x_2)^{-1/2}-(k-x_1)(k+x_1)^{-1/2}+(x_2-1)(x_2+1)^{-1/2}=$$

$$=(k-1)(k+1)^{-1/2}, \quad k \leqslant x_2 \leqslant x_{21}. \tag{2.14}$$

For curve $M_{12}M_{13}$

$$\left.\begin{aligned}(x_1-x_2)(x_1+x_2)^{-1/2}+(x_2-a_1)(x_2+a_1)^{-1/2}+(k-x_1)(k+x_1)^{-1/2}\\=2k(k+1)^{-1/2}-(k+a_1)^{1/2}, \quad x_{112} \leqslant x_1 \leqslant x_{113},\end{aligned}\right\} \tag{2.15}$$

where x_{112} is the solution of equation (2.15) with respect to x_1 where $x_2 = x_{21}$, and x_{113} is the solution of equation (2.11) with respect to x_1 where $x_2 = k$.

For curve $M_{13}M_{14}$

$$\left.\begin{aligned}(x_1-x_2)(x_1+x_2)^{-1/2}-(x_2-a_1)(x_2+a_1)^{-1/2}+(x_1-a_1)(x_1+a_1)^{-1/2}\\=(2a_1)^{1/2}-2a_1(k+a_1)^{-1/2}, \quad x_{113} \leqslant x_1 \leqslant a_2.\end{aligned}\right\} \tag{2.16}$$

For curve MM_{10}

$$x_2=k, \quad 1 \leqslant x_1 \leqslant x_{110}, \tag{2.17}$$

where x_{110} is the solution of equation (2.14) with respect to $x_1 (x_2 = k)$.

For curve M_1M_{12}

$$x_2=x_{21}, \quad 1 \leqslant x_1 \leqslant x_{112}. \tag{2.18}$$

For curve $M_{10}M_{11}$

$$x_2=k, \quad x_{110} \leqslant x_1 \leqslant x_{113}. \tag{2.19}$$

For curve $M_{11}M_{13}$

$$x_1 = x_{113}, \quad k \leqslant x_2 \leqslant x_{213}, \tag{2.20}$$

where x_{213} is the solution of equation (2.15) with respect to x_2 where $x_1 = x_{113}$.

Curves B_1M_{14}, $M_{14}B_3$, B_3B_2, M_2B_2, M_3M_4, $M_4M_5M_5B_1$ belong to sides of square A.

The thus defined curves subdivide square A into areas: curve $MM_1M_2M_3M$ bounds area A_1, curve $MM_3M_4M_6M_7M_8M$ bounds area A_2, /68 curve $M_4M_5M_6M_4$ bounds area A_3, curve $M_7M_9M_8M_7$ bounds area A_4, curve $MM_8M_9M_{11}M_{10}M$ bounds area A_5, curve $M_{10}M_{11}M_{13}M_{12}M_{10}$ bounds area A_6, curve $B_1B_3M_{13}M_{11}M_9M_7M_6M_5B_1$ bounds area A_7, curve $MM_{10}M_{12}$-M_1M bounds area A_9.

After defining the subdivision of square A, we can formulate the theorem of synthesis of optimal control.

Theorem

Any point $\bar{x}$ of square A will transfer to point (1, k) with the least increase of the argument if controls u_1, u_2 are selected as follows:

$$\begin{array}{llll}
\text{if} & \bar{x} \in A_1 \vee A_6 \vee A_7 \vee B_2M \vee M_{14}B_1 \vee M_1M, & \text{then} & u_1 = -1,\ u_2 = 0, \\
\text{if} & \bar{x} \in A_2 \vee M_5M_3 \vee M_9M, & \text{then} & u_1 = +1,\ u_2 = 0, \\
\text{if} & \bar{x} \in A_3 \vee A_5 \vee A_8 \vee A_9 \vee B_1M_5 \vee M_{14}B_3B_2, & \text{then} & u_1 = -1,\ u_2 = +1, \\
\text{if} & \bar{x} \in A_4 \vee M_3M, & \text{then} & u_1 = +1,\ u_2 = +1.
\end{array}$$

Note 1. The theorem is valid under the condition that $k > 1$, a_1 is sufficiently small quantity, and a_2 is sufficiently large.

Note 2. The problem stated in section 1 is tantamount to the problem of this section. In reality, if the final point is defined by the integers x_1^k, $x_2^k (x_1^k < x_2^k)$, then after replacing the variables: $y_1 = (x_1^k)^{-1}$, $y_2 = (x_1^k)^{-1} x_2$, $t' = (x_1^k)^{1/2} t$, we will be under conditions of problem of section 2. If, however $x_1^k > x_2^k$ it is sufficient to change the direction on the apsidal line so that variables x_1, x_2 switch roles.

Note. 3. With sufficiently large k, area A_4 degenerates into a point lying on straight line $x_1 = 1$.

3. Proof of the Theorem of Synthesis

In the theory of optimal control, verification of possible solutions is done with the aid of sufficient conditions of optimacy. The set of such conditions for the problem of speed of response is united in the concept of regular synthesis [2]. But regular synthesis is constructed in an open set, and in our case the set in which the synthesis is constructed (square A) is closed. This fact is essential since a portion of the optimal trajectory can belong to the boundary of A. Thus, let us prove the theorem using the method proposed in study [3].

Let us consider the functions $P_1(x_1, x_2)$ and $P_2(x_1, x_2)$. After designating with P_1^i and P_2^i the values of the functions P_1 and P_2 in area A_i, let us define them as follows:

$$P_1^1 = x_2 (x_1 + x_2)^{-3/2} - 1/2 \int_k^{x_1} (x_1 + y)^{-3/2} dy,$$
$$P_2^1 = -x_1 (x_1 + x_2)^{-3/2}; \tag{3.1}$$

$$P_1^2=x_2(x_1+x_2)^{-3/2}+3/2\int_{x_2}^{k}(x_1-y)(x_1+y)^{-5/2}\,dy,\quad P_2^2=x_1(x_1+x_2)^{-3/2}; \tag{3.2}$$ /69

$$P_1^3=-x_2(x_1+x_2)^{-3/2},\quad P_2^3=x_1(x_1+x_2)^{-3/2}-1/2\int_{a_1}^{k}(x_2+y)^{-3/2}\,dy; \tag{3.3}$$

$$P_1^4=x_2(x_1+x_2)^{-3/2},\quad P_2^4=x_1(x_1+x_2)^{-3/2}+3/2\int_{x_1}^{1}(x_2-y)(x_2+y)^{-5/2}\,dy; \tag{3.4}$$

$$P_1^5=-x_1(x_1+x_2)^{-3/2},\quad P_2^5=x_1(x_1+x_2)^{-3/2}-1/2\int_{1}^{x_1}(x_2+y)^{-3/2}\,dy; \tag{3.5}$$

$$P_1^6=-x_2(x_1+x_2)^{-3/2}+3/2\int_{k}^{x_2}(x_1-y)(x_1+y)^{-5/2}\,dy,\quad P_2^6=-x_1(x_1+x_2)^{-3/2}; \tag{3.6}$$

$$P_1^7=-x_2(x_1+x_2)^{-3/2}+3/2\int_{a_1}^{x_2}(x_1-y)(x_1+y)^{-5/2}\,dy,\quad P_2^7=-x_1(x_1+x_2)^{-3/2}, \tag{3.7}$$

$$P_1^8=-x_2(x_1+x_2)^{-3/2},\quad P_2^8=-x_1(x_1+x_2)^{-3/2}+3/2\int_{a_1}^{x_1}(x_2-y)(x_2+y)^{-5/2}\,dy; \tag{3.8}$$

$$P_1^9=-x_2(x_1+x_2)^{-3/2},\quad P_2^9=-x_1(x_1+x_2)^{-3/2}+3/2\int_{1}^{x_1}(x_2-y)(x_2+y)^{-5/2}\,dy. \tag{3.9}$$

By the clear form of functions P_1 and P_2 defined by formulas (3.1)-(3.9), it is easy to verify that the system whose set of velocities is a line $P_1x_1 + P_2x_2 = 1$ is also (following the terminology of study [3]) auxiliary. The selection of optimal controls is done namely as the theory of synthesis asserts.

The theorem is proven.

REFERENCES

1. Metody optimizatsii s prilozheniyami k mekhanike kosmicheskogo poleta [Methods of optimization with applications to the mechanics of space flight], "Nauka" Publishers, 1965.

2. Boltyanskiy, V. G., Matematicheskiye metody optimal'nogo upravleniya [Mathematical methods of optimal control], "Nauka" Publishers, 1969.

3. Shmyrov, A. S., "Problem of speed of response in a plane," in: Mekhanika upravlyayemogo dvizheniya i problemy kosmicheskoy dinamiki, Ed. by V. S. Novoselov, Leningrad, 1972, p. 57.